Last time: addition & subtraction
& scalar multiplication

Dot product of vectors

Definition

If $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$w = (w_1, \dots, w_n) \in \mathbb{R}^n$,

then $$v \cdot w = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = \sum_{j=1}^{n} v_j w_j \in \mathbb{R}^1$$ scalar.

Other notation: $\langle v, w \rangle = v \cdot w$

Geometric formula for the dot product:

$$v \cdot w = \|v\| \, \|w\| \cos\theta$$

"norm of v" = "length of v"

angle between the two vectors (the small one).

Example: Find $v \cdot w$ and the angle between v & w if

ⓐ $v = (-1, 2)$, $w = (3, -1)$

ⓑ $v = (1, 0, 2, -4, -1)$, $w = (2, 2, -1, -1, 5)$

Answers ⓐ $v \cdot w = (-1)(3) + (2)(-1) = -5.$ ✓

$$\|v\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

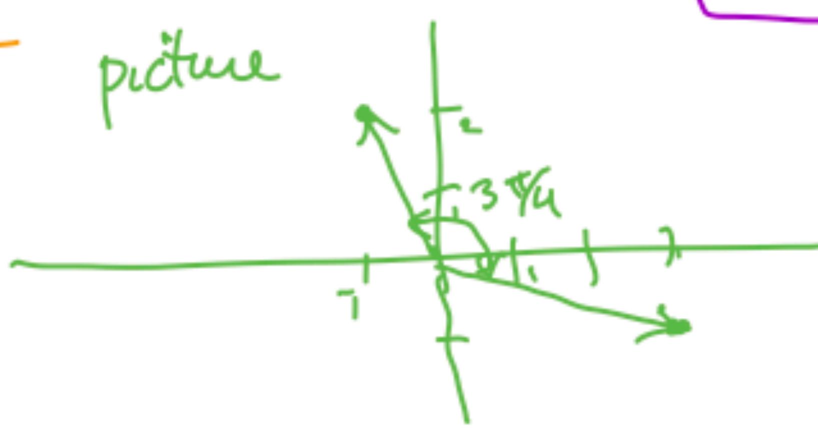
$$\|w\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

Since $v \cdot w = \|v\| \, \|w\| \cos\theta$

$$-5 = \sqrt{5}\,\underbrace{\sqrt{10}}_{\sqrt{2}\sqrt{5}} \cos\theta = \sqrt{2} \cdot 5 \cos\theta$$

$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \boxed{\theta = 135^\circ = \frac{3\pi}{4}}$$

picture



(b) $v = (1,0,2,-4,-1)$, $w = (2,2,-1,-1,5)$

$$v \cdot w = 1\cdot 2 + 0\cdot 2 + 2\cdot(-1) + (-4)(-1) + (-1)5 = \boxed{-1}$$

$$\|v\| = \sqrt{1^2+0^2+2^2+(-4)^2+(-1)^2} = \sqrt{22}$$

$$\|w\| = \sqrt{2^2+2^2+(-1)^2+(-1)^2+5^2} = \sqrt{35}$$

$$\Rightarrow v\cdot w = \|v\|\,\|w\|\cos\theta$$

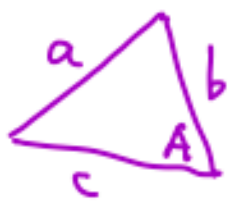
$$-1 = \sqrt{22}\,\sqrt{35}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{-1}{\sqrt{22}\sqrt{35}}$$

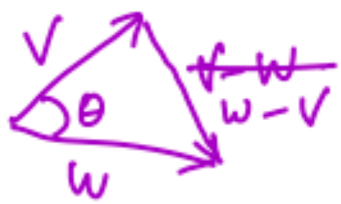
$$\theta = \arccos\left(\frac{-1}{\sqrt{22}\sqrt{35}}\right) \simeq 1.606\ldots$$
$$= 92.06^\circ$$

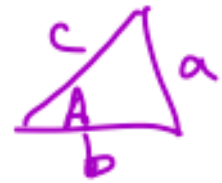
Where does that geometric formula come from?

Answer: Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos A$$





Law of Cosines: $a^2 = b^2 + c^2 - 2bc(\cos A)$

$$\Rightarrow \|w-v\|^2 = \|w\|^2 + \|v\|^2 - 2\|w\|\|v\|\cos\theta$$

Important fact: If $v \in \mathbb{R}^n$, then

$$v \cdot v = \|v\|^2 = \langle v, v\rangle$$

Proof: $v \cdot v = v_1^2 + v_2^2 + \dots + v_n^2 = \left(\sqrt{v_1^2 + \dots + v_n^2}\right)^2$

$= \|v\|^2.$

$$\Rightarrow \langle w-v, w-v\rangle = \langle w,w\rangle + \langle v,v\rangle - 2\|w\|\|v\|\cos\theta$$

$$\Rightarrow \langle w, w-v\rangle - \langle v, w-v\rangle = LHS$$

distributive property works.

$$\Rightarrow \langle w,w\rangle - \langle w,v\rangle - \left(\langle v,w\rangle + \langle v,v\rangle\right) = LHS$$

distr

$$\Rightarrow \langle w,w\rangle \underbrace{- \langle w,v\rangle - \langle v,w\rangle}_{\langle v,w\rangle = \langle w,v\rangle} - \langle v,v\rangle = LHS = RHS$$

$$\Rightarrow \langle w,w\rangle - 2\langle v,w\rangle + \langle v,v\rangle = LHS$$

$$\Rightarrow \quad LHS = RHS$$

$$\cancel{\langle w,w\rangle} - 2\langle v,w\rangle + \cancel{\langle v,v\rangle} = \cancel{\langle w,w\rangle} + \cancel{\langle v,v\rangle} - 2\|w\|\|v\|\cos\theta$$

$$\Rightarrow -2\langle v,w\rangle = -2\|w\|\|v\|\cos\theta$$

$$\Rightarrow \boxed{\langle v,w\rangle = \|v\|\,\|w\|\cos\theta} \quad \checkmark \quad \square$$

Properties of vector addition, subtraction, norms, dot products.

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① Commutative Properties

$\forall\, v,w \in \mathbb{R}^n$,

$$v+w = w+v$$

$$v\cdot w = w\cdot v$$

② Associative Property

$\forall\, v,w,u \in \mathbb{R}^n$,

$$(v+w)+u = v+(w+u)$$

[Note $(v\cdot w)\cdot u$ would not make sense, because $v\cdot w \in \mathbb{R}^1$, $u \in \mathbb{R}^n$.]

③ Distributive Properties

$\forall\, v, w, u \in \mathbb{R}^n$, $\quad c \in \mathbb{R}$, $k \in \mathbb{R}$

$c(v+w) = cv + cw$ (scalar mult)

$(c+k)v = cv + kv$

$c(v-w) = cv - cw$

$(c-k)v = cv - kv$

$(v \pm w) \cdot u = v \cdot u \pm w \cdot u$

$u \cdot (v \pm w) = u \cdot v \pm u \cdot w$

④ Norm properties

- $\|v\|^2 = \langle v, v \rangle \geq 0$,
  $= 0$ only if $v = 0$ vector $= (0, 0, \ldots)$
- $\|cv\| = |c| \cdot \|v\|$
- $\|v+w\| \leq \|v\| + \|w\|$

(called triangle inequality:

)

= occurs only if v & w are in same direction

- $\|v-w\| \geq \big|\|v\|-\|w\|\big|$

other triangle inequality.

Another important fact about the dot product:

If $v,w \in \mathbb{R}^n$, then

$$v \cdot w = 0 \iff v \perp w$$

"if and only if" "v is perpendicular (orthogonal) to w."

zero vector is defined to be orthogonal to every vector.

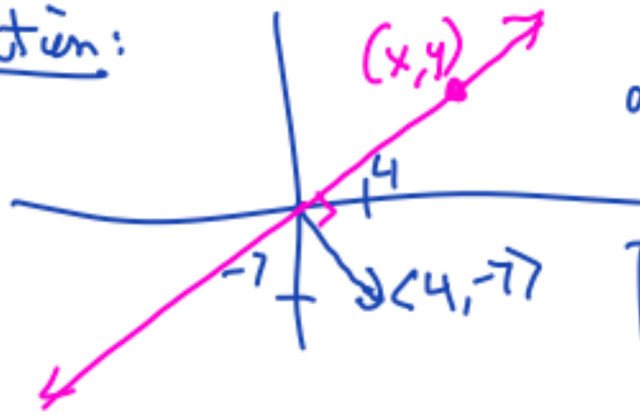
Why is this true?

$$v \cdot w = 0 \iff \|v\|\,\|w\|\cos\theta = 0$$

$$\iff \theta = 90^\circ.$$

Example: Find the equation of the line through the origin in $\mathbb{R}^2$ that is perpendicular to $4\hat{i} - 7\hat{j}$.

Solution:

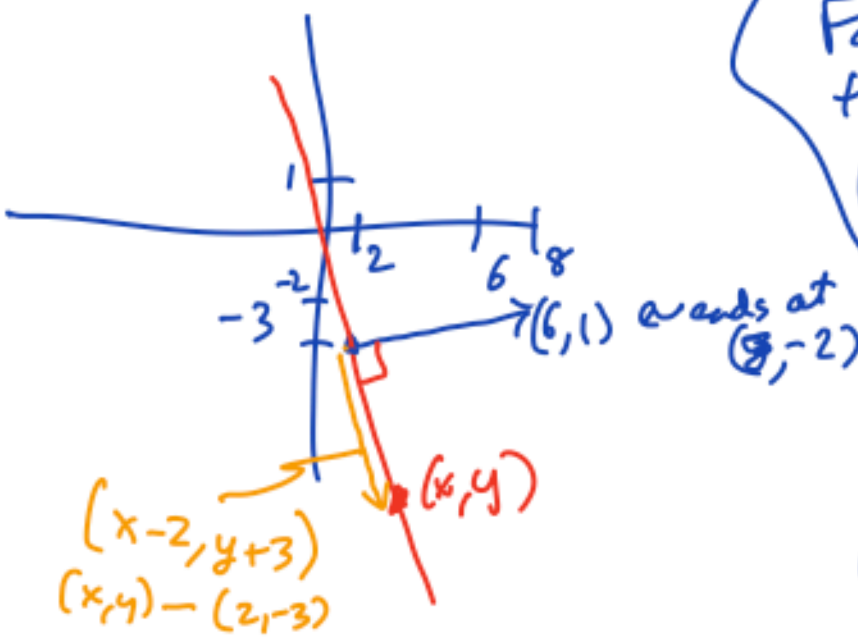


For every (x,y) on this line,

$$(x,y)\cdot(4,-7)=0$$

$$\boxed{4x-7y=0}.$$

Example Find the equation of the line through $(2,-3)$ that is perpendicular to the vector $(6,1)$.



For all (x,y) on the line:

$$(6,1)\cdot(x-2,y+3)=0$$

$\Rightarrow$

$$6(x-2)+1(y+3)=0$$

$$6x-12+y+3=0$$

$$\boxed{6x+y-9=0}$$

Example: Find the equation of the plane in $\mathbb{R}^3$ that contains $(1,1,1)$ and is $\perp$ to $(2,9,3)$.

(x,y,z) is in this plane

$$\iff (2,9,3)\cdot(x-1,y-1,z-1) = 0$$

$$2(x-1)+9(y-1)+3(z-1)=0$$

$$2x-2+9y-9+3z-3=0$$

$$\boxed{2x+9y+3z-14=0}$$

And next we could find the equation of a 23-dimensional hyper plane in $\mathbb{R}^{24}$.